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Supplementary Note on Projective Invariants.

By E. O. Lovett.

While a note on projective invariants, inserted in the April number of volume XXI of this Journal was printing, Lie's method for determining differential invariants was applied to the construction* of the following more general second order differential projective invariants of a system of $m+1$ points in a space of $n+1$ dimensions, namely,

$$\sum_{0}^{m}{}_{i} \frac{\left|p^{(k)}_{1,1}, p^{(k)}_{2,2}, \ldots\ldots, p^{(k)}_{n,n}\right|}{\left|p^{(i)}_{1,1}, p^{(i)}_{2,2}, \ldots\ldots, p^{(i)}_{n,n}\right|} \left\{ \frac{x_0^{(i)} - x_0^{(k)} - \sum_{1}^{n}{}_{j}\, p_j^{(i)} \left(x_j^{(i)} - x_j^{(k)}\right)}{x_0^{(i)} - x_0^{(k)} - \sum_{1}^{n}{}_{j}\, p_j^{(k)} \left(x_j^{(i)} - x_j^{(k)}\right)} \right\}^{n+2}, \tag{1}$$

$$k = 0, 1, 2, \ldots\ldots, m,$$

where

$$x_0^{(i)},\ x_1^{(i)},\ x_2^{(i)}, \ldots\ldots, x_n^{(i)}, \qquad i = 0, 1, 2, \ldots\ldots, m \tag{2}$$

are the coordinates of the system of points, and

$$p_j^{(i)} = \frac{\partial x_0^{(i)}}{\partial x_j}, \quad p_{i,j} = \frac{\partial^2 x_0^{(i)}}{\partial x_i\, \partial x_j} = p_{j,i}. \tag{3}$$

For the case $m = 1$, the invariants reduce to the single one

$$\frac{\left|p^{(k)}_{1,1}, \ldots\ldots, p^{(k)}_{n,n}\right|}{\left|p^{(i)}_{1,1}, \ldots\ldots, p^{(i)}_{n,n}\right|} \left\{ \frac{x_0^{(i)} - x_0^{(k)} - \sum_{1}^{n}{}_{j}\, p_j^{(i)} \left(x_j^{(i)} - x_j^{(k)}\right)}{x_0^{(i)} - x_0^{(k)} - \sum_{1}^{n}{}_{j}\, p_j^{(k)} \left(x_j^{(i)} - x_j^{(k)}\right)} \right\}^{n+2}; \tag{4}$$

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* See Comptes Rendus, August 16, 1898; Darboux's Bulletin des sciences mathématiques, January, 1899.

moreover, the invariance of the forms (1) follows *a fortiori* by summation from the invariance of (4).

In the note referred to above, the cases of (1) corresponding to $m = r$, $n = 1, 2$ were applied to the construction of certain geometrical theorems.

The particular cases of (4) corresponding to $m = 1$, $n = 1, 2$ were given by Mr. Bouton;* as remarked for the general case, the invariance of (1) for $m = r$, $n = 1, 2$ follows *a fortiori* from the invariance of the form (4) corresponding to $m = 1, n = 1, 2$.

By noting the relation†

$$(-1)^n |p^{(i)}_{1,1}, \ldots\ldots, p^{(i)}_{n,n}| = G_i \left\{ 1 + \sum_1^n {}_j (p^{(i)}_j)^2 \right\}^{\frac{n}{2}+1}, \tag{5}$$

where G_i is the Gaussian curvature as generalized by Kronecker,‡ the above invariants may be given the following geometrical interpretation:

Take $m + 1$ hypersurfaces arbitrarily chosen, except that through every one of the $m + 1$ points of the system, there should pass at least one hypersurface; let P_i, P_k, H_i, H_k be respectively the points $(x^{(i)}_0, \ldots\ldots, x^{(i)}_n)$, $(x^{(k)}_0, \ldots\ldots, x^{(k)}_n)$, and the hypersurfaces through these points; join P_k by straight lines to all the other points of the system; let θ_i be the angle between the normal to H_k at P_k and the straight line $P_i P_k$, ϕ_i the angle between the line $P_i P_k$ and the normal to H_i at P_i; finally, let $\rho_{j,1}, \rho_{j,2}, \ldots\ldots, \rho_{j,n}$ be the principal radii of curvature of H_j at P_j; then the invariance of the expressions (1) shows that the forms

$$\sum_0^m {}_i \left\{ \prod_{j=1}^{j=n} \rho_{k,j} \cos^{n+2} \theta_i \Big/ \prod_{j=1}^{j=n} \rho_{i,j} \cos^{n+2} \phi_i \right\}, \tag{6}$$
$$k = 0, 1, 2, \ldots\ldots, m$$

suffer no change by projective transformation; and in particular it follows from the invariance of (4) that every element of the last summation is an absolute constant.

By employing the investigations§ of Casorati and von Lilienthal in the theory of curvature, the geometric interpretation of the above invariants can be varied; these latter forms are simple enough for space of three dimensions, but become very complicated for spaces of higher dimensions; this fact is another confirmation of Darboux's dictum,* that the total curvature is the most important notion of curvature for geometry.

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* Bulletin of the American Mathematical Society, April, 1898.

† Beez, Mathematische Annalen, Bd. VII.

‡ Berichte der Berliner Akademie der Wissenschaften, 1869.

§ Acta Mathematica, tomes XIV, XVI.

The invariants (1) and (4) are productive of various geometrical theorems for which the reader is referred to the notes already cited.

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* "Théorie des Surfaces," t. II, p. 365.

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